Week N: Interaction Models

POLI502

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University of South Carolina

Outline for Today

Interaction models

1. Numerical $\times$ binary variable
   - Estimation in R
   - Plot
2. Numerical $\times$ numerical
   - Estimation in R
   - Plot

Review: multiple regression

Multiple regression (additive model):

$$\hat{Y} = \alpha + \beta_1 * X + \beta_2 * Z$$

where Z is a binary variable (0 or 1)

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The model allows us to have different intercepts depending on Z, but the slope for X (β_1) is **assumed to be the same**.

We may want to relax this assumption.

Conditional hypotheses

We might have a third variable (Z) that not only influences the effect of X but also **conditions** it.

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Instead, we want to have this (interaction model):

$$\hat{Y} = \alpha + \beta_1 * X + \beta_2 * Z + \beta_3 * \mathbf{XZ}$$
$$\hat{Y} = \alpha + \beta_1 * X \text{ when Z=0}$$
$$\hat{Y} = (\alpha + \beta_2) + (\beta_1 + \beta_3) * X \text{ when Z=1}$$

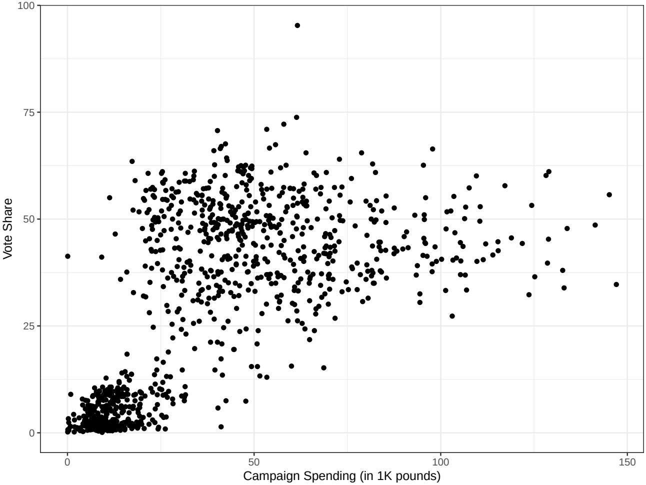
Now, both the intercept and the slope are different

Example 1: Effect of campaign spending

In competitive elections, the more money a candidate spends on campaigning, the more votes s/he is expected to get.

- DV: vote share (%) for candidates in the 2009 general election in Japan
- IDV: campaign spending (in 1,000 pounds)

Example 1: Effect of campaign spending



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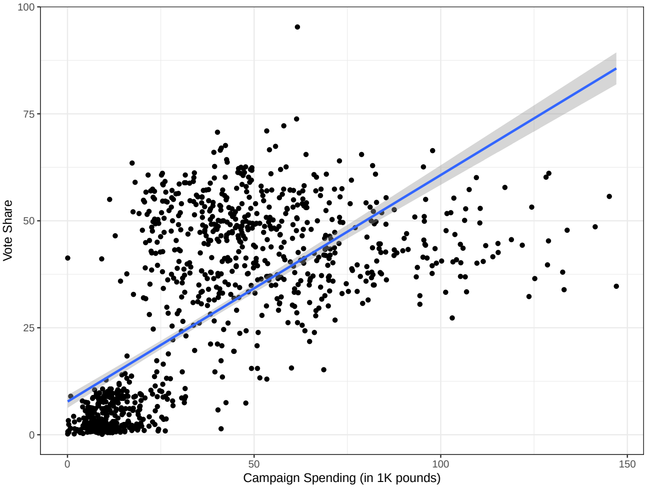

Example 1: Effect of campaign spending

Table:

	Dependent variable:
	Vote Share
Campaign Spending	0.530***
	(0.017)
Constant	7.735***
	(0.757)
Observations	1,124
R^2	0.478
Adjusted R^2	0.478
Residual Std. Error	16.042 (df = 1122)
Note:	*p<0.1; **p<0.05; ***p<0.01

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$$\hat{VS} = 7.735 + 0.53 * CS$$

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Four hurdles to clear:

1. Causal mechanism linking $X \Rightarrow Y$
2. No reverse causality $Y \Rightarrow X$
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Four hurdles to clear:

1. Causal mechanism linking $X \Rightarrow Y$
2. No reverse causality $Y \Rightarrow X$
3. X and Y covary
4. No confounding Z

What do you think the causal mechanism is here?

What do you think potential confounders (conditioning factors) are?

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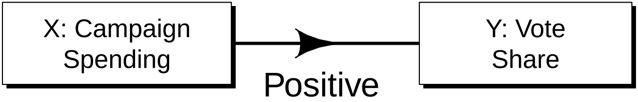

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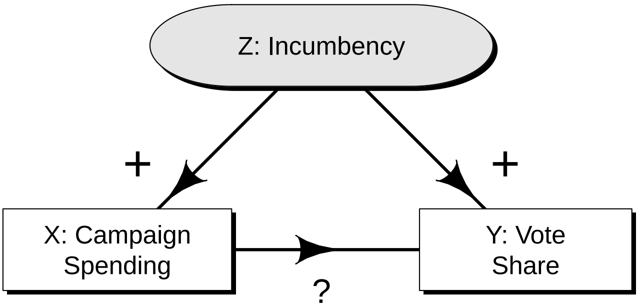

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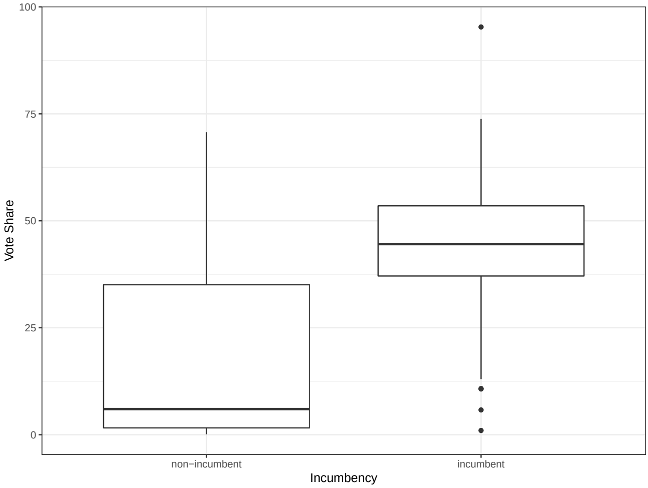

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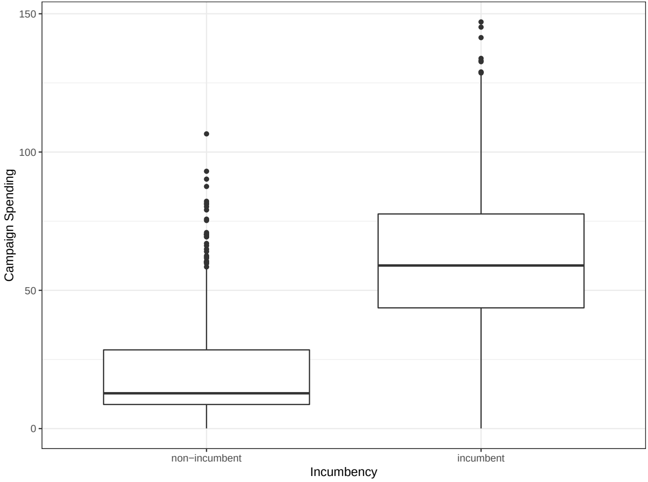

Example 1: Effect of campaign spending

Table:

	(1)	(2)
Campaign Spending	0.530***	0.391***
	(0.017)	(0.022)
Incumbent		12.238***
		(1.335)
Constant	7.735***	8.384***
	(0.757)	(0.734)
Observations	1,124	1,124
R^2	0.478	0.515
Adjusted R^2	0.478	0.514
Residual Std. Error	16.042 (df = 1122)	15.480 (df = 1121)
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$$\hat{VS} = 8.384 + 0.391 * CS + 12.238 * Inc$$

$$\hat{VS} = 8.384 + 0.391 * CS \text{ when Z=0 } (1)$$

$$\hat{VS} = 20.622 + 0.391 * CS \text{ when Z=1 } (2)$$

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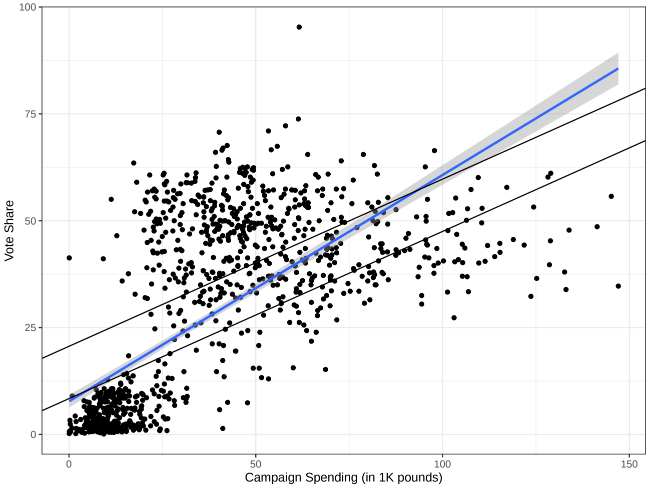

Additive model: different intercept (when z= 0 or 1)

Example 1: Effect of campaign spending

The first model:

$$\hat{VS} = 7.735 + 0.530 * CS$$

The second model:

$$\hat{VS} = 8.384 + 0.391 * CS \text{ for non-incumbents}$$
$$\hat{VS} = 20.622 + 0.391 * CS \text{ for incumbents}$$

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The second model:

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$$\hat{VS} = 20.622 + 0.391 * CS \text{ for incumbents}$$

The second model is more flexible.

- the first one is based on an assumption that incumbents and non-incumbents have the same intercept;
- the second one relaxes that assumption.

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- If the effect of Campaign Spending on Vote Share is *through* increased publicity, the effect could be bigger for non-incumbents
    - Campaign spending may increase vote share in general;

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  - Campaign spending may increase vote share in general;
  - Yet, an additional spending will increase vote share more for non-incumbents;
    An additional spending may have little effect on vote share for incumbents, as they are relatively well known already;

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- If the effect of Campaign Spending on Vote Share is *through* increased publicity, the effect could be bigger for non-incumbents
    - Campaign spending may increase vote share in general;
    - Yet, an additional spending will increase vote share more for non-incumbents;
      An additional spending may have little effect on vote share for incumbents, as they are relatively well known already;
    - → depending on the incumbency status, not only the intercept but also the slope for spending may differ.

Example 1: Effect of campaign spending

To relax this assumption, we include a **product** of Campaign Spending and Incumbency status:

$$\hat{VS} = \alpha + \beta_1 * CS + \beta_2 * Inc + \underline{\beta_3 * CS * Inc}$$

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- When $Inc = 0$, the model simplifies to

$$\begin{aligned} \hat{VS} &= \alpha + \beta_1 * CS + \beta_2 * (0) + \beta_3 * CS * (0) \\ &= \alpha + \beta_1 * CS \end{aligned}$$

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- When $Inc = 1$, the model reduces to

$$\begin{aligned} \hat{VS} &= \alpha + \beta_1 * CS + \beta_2 * (1) + \beta_3 * CS * (1) \\ &= \alpha + \beta_1 * CS + \beta_2 + \beta_3 * CS \\ &= (\alpha + \beta_2) + (\beta_1 + \beta_3) * CS \end{aligned}$$

Example 1: Effect of campaign spending

Table:

	(1)	(2)	(3)
Campaign Spending	0.530***	0.391***	0.864***
	(0.017)	(0.022)	(0.026)
Incumbency		12.238***	46.758***
		(1.335)	(1.791)
Campaign Spending × Incumbency			−0.866***
			(0.036)
Constant	7.735***	8.384***	−1.502**
	(0.757)	(0.734)	(0.721)
Observations	1,124	1,124	1,124
R^2	0.478	0.515	0.681
Adjusted R^2	0.478	0.514	0.680
Residual Std. Error	16.042 (df = 1122)	15.480 (df = 1121)	12.548 (df = 1120)
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$$\hat{VS} = -1.502 + 0.864 * CS + 46.758 * Inc - 0.866 * CS * Inc$$

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- When $Inc = 0$, the model simplifies to

$$\hat{VS} = -1.502 + 0.864 * CS$$

- When $Inc = 1$, the model simplifies to

$$\begin{aligned}\hat{VS} &= -1.502 + 0.864 * CS + 46.758 - 0.866 * CS \\ &= (-1.502 + 46.758) + (0.864 - 0.866) * CS \\ &= 45.256 - 0.002 * CS\end{aligned}$$

Estimation in R

When you want to include an interaction term between `x` and `z`, you write

```
lm(y ~ x + z + x*z)
```

(The third term that combines `x` and `z` with a colon or asterisk ($X \star Z$) is the interaction term.)

Whenever you estimate an interactive model, make sure you interpret the results graphically using the effect function.

```
plot(effect(term = "x:z", mod = fit))
```

Or even better, use ggplot to plot the effect plots.

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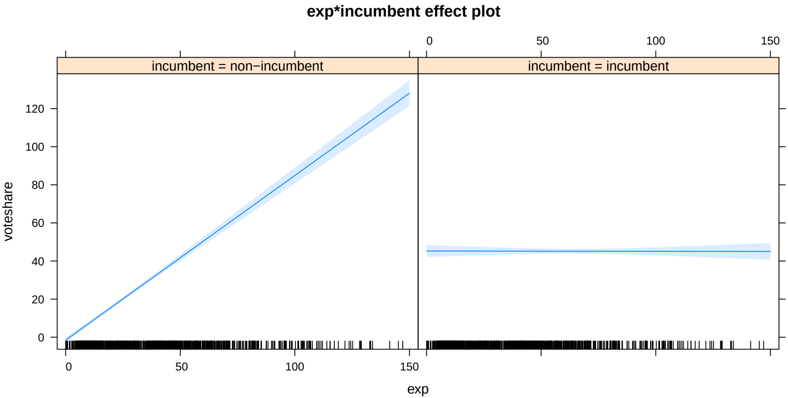

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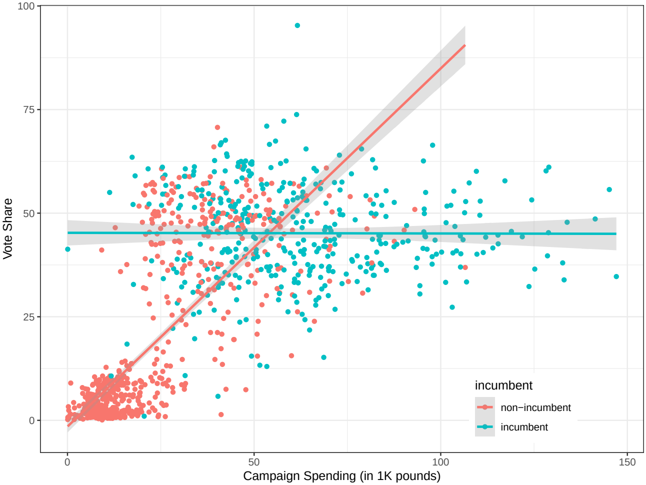

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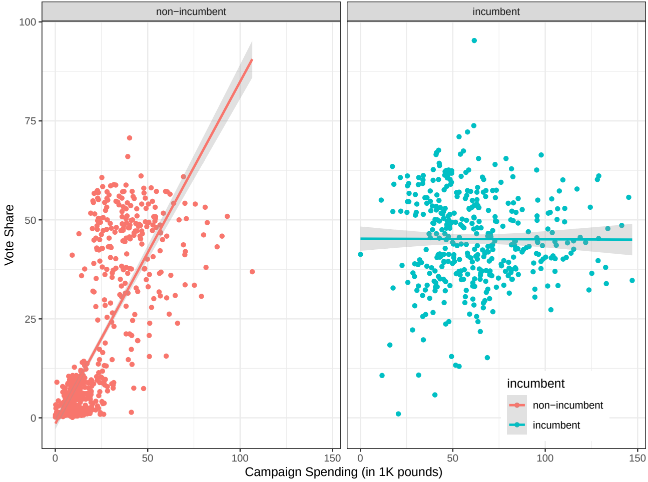


Estimation in R: factor vs numeric

Recall that, when including a binary variable in a regression model, we could do so in one of two ways.

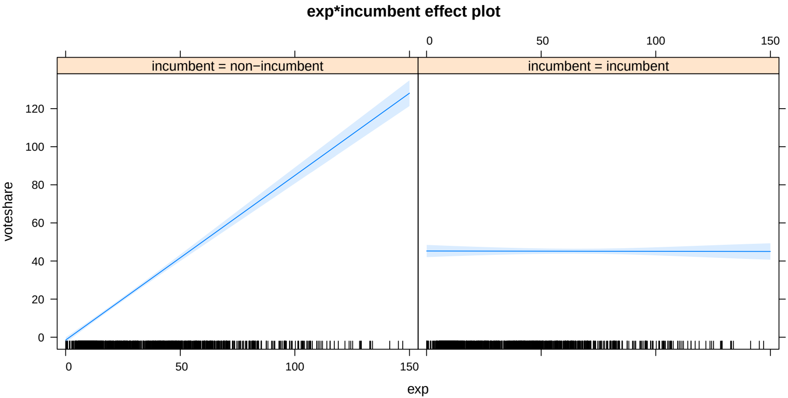
- Include the original factor variable as is
- Include a numerical binary variable

We saw this when dealing with the NorthSouth binary in the Putnam data set:

- Include the original factor variable is necessary to create an effect "plot" (o.w. R will assume it is a numeric variable)
- Include a numerical binary variable is preferable in order to produce an intuitive regression "table"

The same applies here.

Example 1: factor vs numeric



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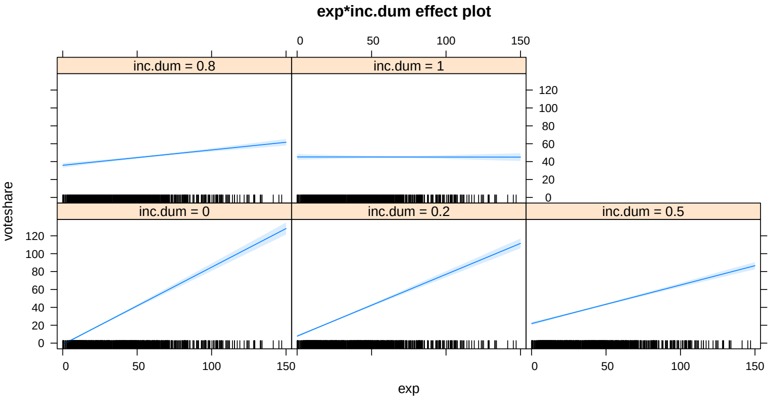

Splitting the sample

There is another way to obtain a similar regression results, **but you need to be careful about sample size**:

- Split the data into two subsets (incumbents and non-incumbents);
- Regress Vote Share on Campaign Spending on each subset.

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- Split the data into two subsets (incumbents and non-incumbents);
- Regress Vote Share on Campaign Spending on each subset.

	(1)	(2)
Campaign Spending	-0.002	0.864^{***}
	(0.023)	(0.027)
Constant	45.256^{***}	-1.502^{**}
	(1.558)	(0.740)
Observations	392	732

- Results for $Inc = 0$

$$\hat{VS} = -1.502 + 0.864 * CS$$

- Results for $Inc = 1$

$$\hat{VS} = 45.256 - 0.002 * CS$$

Interaction terms Do's and Don'ts

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$$\hat{VS} = \alpha + \beta_1 * CS + \beta_2 * Inc + \beta_3 * CS * Inc$$

- What happens if we (mistakenly) drop $\beta_2 * Inc$...?

Interaction terms Do's and Don'ts

(2) Caveat: **The numerical results (coefficients) in interaction models can often be misleading**; always interpret the results graphically (draw implied regression lines)

- Just because you get a statistically significant coefficient on an interaction term, it does NOT automatically mean that you find a meaningful conditional relationship! $\rightarrow$ look at the slope/effect size
- Likewise, even if your interaction term is statistically significant, it does NOT mean that your conditional relationship will NOT be substantial!

(3) Z (modifying variable) does not have to be a binary variable. It can be a continuous variable as well.

Example 2: Female representation

Let's say we want to test the following hypotheses:

- As the level of ethnic fractionalization increases, female representation goes down.

  This relationship may not hold universally;

  - The above relationship will not exist in poorer countries;
  - The relationship will be stronger in wealthier countries.

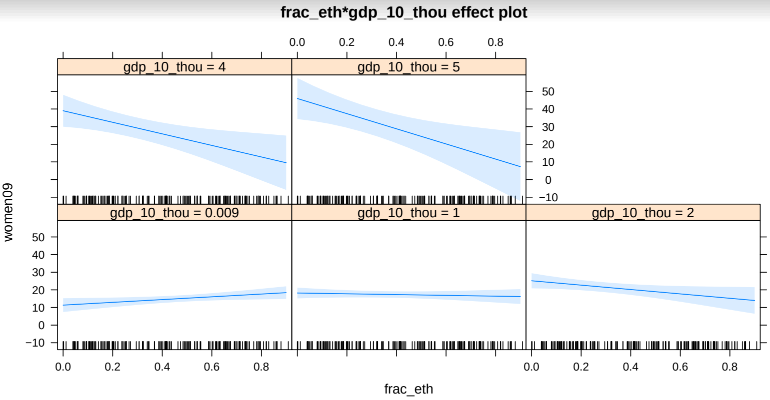
To test these hypotheses, we could estimate the following model:

```
lm(women09 ~ frac_eth + gdp_10_thou +
frac_eth*gdp_10_thou)
```

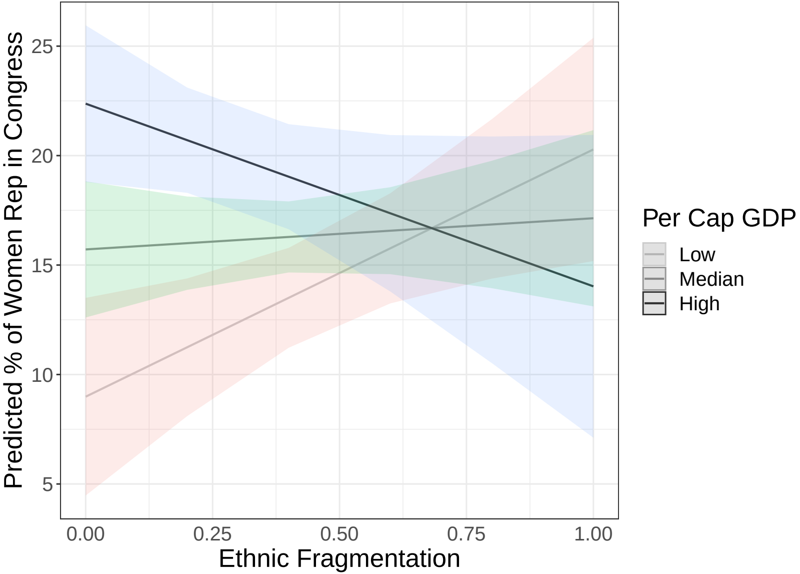
Example 2: Female representation

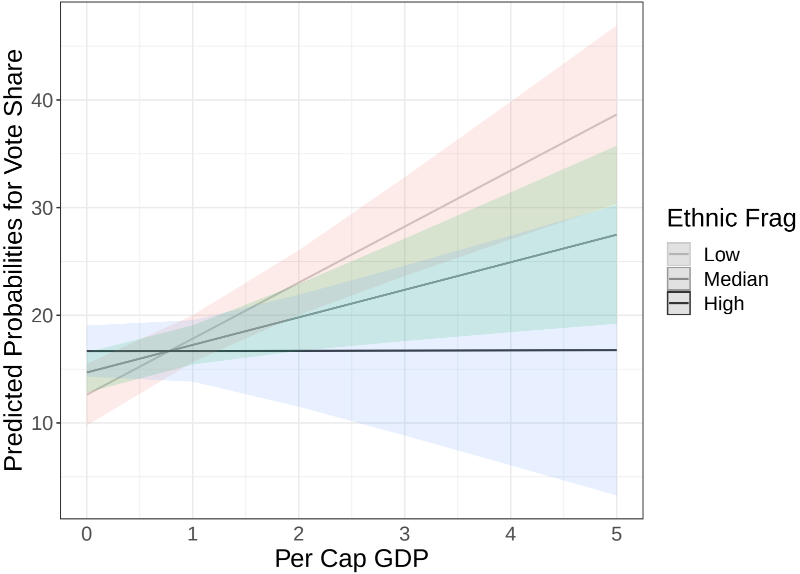
Table:

	(1)	(2)
Ethnic Fractionalization	2.406	7.937**
	(3.300)	(3.697)
Per capita GDP	3.610***	6.936***
	(0.880)	(1.390)
Ethnic Fractionalization × Per capita GDP		−10.177***
		(3.344)
Constant	13.776***	11.275***
	(1.855)	(1.987)
Observations	166	166
Adjusted R^2	0.084	0.128
Residual Std. Error	10.359 (df = 163)	10.106 (df = 162)
Note:		*p<0.1; **p<0.05; ***p<0.01



frac_eth*gdp_10_thou effect plot
gdp_10_thou = 4
gdp_10_thou = 5
gdp_10_thou = 0.009
gdp_10_thou = 1
gdp_10_thou = 2
women09
frac_eth
0.0
0.2
0.4
0.6
0.8
50
40
30
20
10
0
−10


Predicted % of Women Rep in Congress
25
20
15
10
5
0.00
0.25
0.50
0.75
1.00
Ethnic Fragmentation
Per Cap GDP
Low
Median
High


Predicted Probabilities for Vote Share
40
30
20
10
0
1
2
3
4
5
Per Cap GDP
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Median
High

Final Advice: all interactions are "symmetric"

When you propose a conditional hypothesis/theory, you actually should test this following pair of hypotheses:

- $H_{X|Z}$: The marginal effect of X on Y is positive at all values of Z; this effect is strongest when Z is at its lowest and declines in magnitude as Z increases.
- $H_{Z|X}$: The marginal effect of Z on Y is positive at all values of X; this effect is strongest when X is at its lowest and declines in magnitude as X increases.